Chapter 16

The Role of Presentational Structures in Understanding and Solving Mathematical Word Problems

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Within a theoretical perspective of a cognitive simulation model (Kintsch & Greeno, 1985; Reusser, 1985, 1989b, 1990) we argue that the linguistically cued representation of the situation denoted in a text must be viewed as a crucial step for the successful understanding and solving of word problems. Based on theoretical considerations and empirical data, we suggest that teachers and researchers in mathematics education should become more aware of characteristics of situational and presentational structures of word problems. We suggest further that students would be better helped by focusing attention on characteristics of the episodic situation and problem structure and on how they are presented in the text, rather than on matching linguistic cues or keywords to formal mathematical structures. We conclude that the situational structures to be mathematized are a central source of problem difficulty. Moreover, an analysis of the content-related situational structure of a mathematical word problem makes it possible to examine more precisely the impact of linguistic variations, which may be used to represent specific situational structures. By concentrating on the presentational structure of mathematical word problems, we argue for an integration of different research approaches within a perspective motivated by pedagogical and instructional questions about the purpose and the objectives of using arithmetic word problems in school settings. (For a discussion into similar issues in solving *algebra* word problems, see Weaver & Kintsch, 1992.)

FROM COMPUTATIONAL AND LINGUISTIC FACTORS TO LOGICO-MATHEMATICAL FACTORS

Early Studies of Computational and Linguistic Factors

Early studies of word problem solving have investigated variables, such as type and number of arithmetic operations, problem length (number of words), order in which numbers are presented in the problem text, syntactical complexity, and cue words (e.g., Jerman, 1973; Jerman & Mirman, 1974; Searle, Lorton, & Suppes, 1974). Even though these studies demonstrate that many different variables may influence the level of difficulty of a particular word problem, they provide a rather inconsistent picture of the variables accounting for the variance in the proportion of correct answers to elementary arithmetic word problems.

In these early studies, the dominant theoretical conceptualization of the processes involved in solving arithmetic word problems was that of a *direct translation* of the linguistic surface structure into an appropriate equation (see Bobrow, 1964, for an early computer model for this kind of theory). A point in case for this conceptualization is the use of terms, such as *verbal cues* and *key words*, which were considered linguistic factors that determine the relative difficulty of verbal arithmetic problems (e.g., Jerman, 1973; Searle et al., 1974). Verbal cues—such as "equal," "altogether," "gained," "left," "lost," or "each"—are assumed to signal specific arithmetic operations. Within this approach verbal cues thus presuppose a straightforward matching of specific terms to corresponding mathematical operations. Obviously, such a direct matching only works for very restricted contexts, because the semantically adequate mathematical meaning of any cue depends on a broad variety of factors contributing to the semantic structure of the problem text and its question. The (mis)conception that underlies the term *verbal cue* may be, in part, the result of an artificial mode of presenting word problems in school settings, which makes use of a specific and limited vocabulary. Indeed, students are very easily misled when using such cue words in tasks where a superficial—that is, a purely local—interpretation of the term leads to a wrong arithmetic operation (Nesher & Teubal, 1975). From a theoretical point of view, a genuine solution of a verbal problem is hardly possible by direct translation of its verbal formulation into the mathematical equation.

Information-Processing Models for One-Step Addition and Subtraction Word Problems Focusing on Logico-Mathematical Factors

Over the past dozen years research on understanding and solving arithmetic word problems has been dominated by one very specific problem set for which detailed computational processing models have been constructed. The set of problems referred to are simple one-step addition and subtraction problems that have been constructed within the conceptual framework of three categories of basic semantic structures: *combine, change*, and *compare* structures (Nesher, Greeno, & Riley, 1982; Riley, 1981; Vergnaud, 1982). Such parsimoniously worded problems corresponding to one of these problem types will be referred to as *standard problems*. Examples are shown in Table 16.1.

Combine tasks involve static relationships between sets and ask for a union set or for one of two disjoint subsets. Change problems describe an increase or decrease of some initial state that results in a final state. Compare problems involve comparisons between two static sets, asking either for the difference set or for one of the two sets when the difference set is given. Studies have shown that the overall pattern of relative problem difficulty for these problem types is quite stable (Nesher et al., 1982; Riley & Greeno, 1988). On the average, change problems are the least difficult, compare problems are the most difficult, and combine problems are in between.

Each of the three semantic problem types can be further differentiated, depending on which set corresponds to the identity of the unknown to be determined quantitatively. Within each of the three semantic categories of combine, change, and compare problems, depending, again, on the identity of the unknown, the

TABLE 16.1
Examples of the Standard Problems

Combine Problems
Mary has 3 marbles. Peter has 5 marbles. How many marbles do they have altogether?
Heidi and John have 8 marbles altogether. John has 3 marbles. How many marbles does Heidi have?
Change Problems
Change 1:
Mary had 3 marbles. Then John gave her 5 marbles. How many marbles does Mary have now?
Change 2:
Mary had 6 marbles. Then she gave 4 marbles to John. How many marbles does Mary have now?
Change 3:
Mary had 2 marbles. Then John gave her some marbles. Now Mary has 9 marbles. How many marbles did John give to her?
Change 4:
Mary had 8 marbles. Then she gave some marbles to John. Now Mary has 3 marbles. How many marbles did she give to John?
Change 5:
Mary had some marbles. Then John gave her 3 marbles. Now Mary has 5 marbles. How many marbles did Mary have in the beginning?
Change 6:
Mary had some marbles. Then she gave 2 marbles to John. Now Mary has 6 marbles. How many marbles did she have in the beginning?
Compare Problems
Diana has 5 marbles. Tom has 8 marbles. How many marbles does Tom have more than Diana?
Mary has 9 marbles. She has 4 marbles more than John. How many marbles does John have?

Note. All subtypes are listed only for the category of change problems.

relative difficulty shows rather stable data patterns across several studies. For example, for change problems it has repeatedly been shown that asking for an unknown resulting state (Change 1: after increasing a specific set; Change 2: after decreasing a specific set) are the least difficult problem types. Somewhat more difficult are problems asking for the amount of change or transfer to be determined (Change 3: increasing; Change 4: decreasing). The most difficult are problems with an unknown initial state (Change 5: before increasing; Change 6: before decreasing) (Riley & Greeno, 1988; Riley, Greeno, & Heller, 1983).

Working with parsimoniously worded standard problems within the paradigm of the logico-mathematical categories of combine, change, and compare problems leads to rather consistent data pattern. Several simulation models embodying explicit hypotheses about knowledge structures and processes necessary to solve such standard problems have been worked out in great detail (Briars & Larkin, 1984; Kintsch & Greeno, 1985; Riley & Greeno, 1988; Riley, Greeno, & Heller, 1983) and have provided coherent theoretical explanations.

Most of these models on understanding and solving mathematical word problems, however, deal with the process of mathematization as a process of a more or less direct translation from a textual structure into a logico-mathematical representation. As developmental models (Briars & Larkin, 1984; Riley et al., 1983), they do not deal with language or situation comprehension as sources of word problem difficulty, but rather with the development of *logico-mathematical schemata*.

Only the model of Kintsch and Greeno (1985), whose main objective is to study "the interaction between comprehension and problem solving" (p. 109), includes a thorough analysis of text comprehension processes based on the theory developed by van Dijk and Kintsch (1983). Understanding and solving a word problem is described as a strategic process of constructing a dual representation, including a *propositional textbase* organized around interrelated sets of objects, and an *abstract situation* or *problem model* inferred from the reader's knowledge.

The limitation of the Kintsch and Greeno model, however, lies in an a priori mapping between the propositional structures of the textbase and the set structure of an abstract problem model. The model shows no attempt to explicitly understand the action or situation described by a word problem: The dual representation model jumps directly, in a one-step mathematization process, from the propositional textbase to a set theoretic representation of the problem by applying powerful, cue-word driven arithmetic comprehension strategies.

EFFECTS OF "WORDING" THAT QUESTION THE VALIDITY OF PROCESS MODELS BASED ON LOGICO-MATHEMATICAL STRUCTURES

Even though stable data patterns have been replicated for the parsimoniously worded standard problems, there are also intriguing empirical findings generated by using minor variations of the standard problems that do not fit into the pattern of problem difficulty as presented earlier. Some of these are discussed in the following.

Hudson (1983)

In an often-cited study, Hudson had children from nursery school through first grade solve compare problems that asked for the unknown difference set. Subjects were presented with a card depicting two disjoint sets of objects, differing in cardinality by one, two, or three. The drawings on the cards symbolized such entities as children and cookies, or birds and worms. The following are two versions of the verbal formulation of the problem, which—in terms of the classification of Nesher et al. (1982)—did not differ in their underlying logico-semantic structure and the identity of the unknown quantity asked for:

1. The "How Many More" task: In this condition each child was shown one of the drawings with the two sets of objects. As each drawing was presented, the experimenter pointed to the icons on the card and asked for example: "Here are some birds and here are some worms. How many more birds than worms are there?"

2. The "Won't Get" task: Under this condition the wording of the questions was as follows: "Here are some birds and here are some worms. Suppose the birds all race over, and each one tries to get a worm. Will every bird get a worm? How many birds won't get a worm?"

Under Condition 1, children showed poor performance, but they improved dramatically under Condition 2. Hudson interpreted the children's apparent lack of quantitative reasoning when confronted with problem questions following comparative constructions such as "How many ⟨comparative term⟩ . . . than . . . ?" as having to be accounted for by "a linguistic factor." According to Hudson, it is the children's limited comprehension of such linguistic constructions as "How many more . . . than" that leads to misinterpretations and hence to wrong numerical answers.

De Corte, Verschaffel, and De Win (1985)

These authors investigated for a set of six word problems (corresponding to the three basic semantic categories as conceptualized by Riley et al.) how rewording these problems affects their solution. For example, for a Change 5 problem the solutions of the following versions were compared (rewordings shown in italics):

1. Joe won 3 marbles.
   Now he has 5 marbles.
   How many marbles did Joe have in the beginning?

2. *Joe had some marbles.*
   He won 3 *more* marbles.
   Now he has five marbles.
   How many marbles did Joe have in the beginning?

First graders produced only 13% correct responses for problems presented as in Version 1 as compared to 33% correct responses for problems formulated as in Version 2. For second graders corresponding percentages are: 61% for Version 1 as opposed to 79% for Version 2. Explicitly stating the initial state and the direction of change in relation to the initial state proved to be helpful for the children to arrive at an appropriate solution of the task. De Corte et al. (1985, p. 469) gave the following interpretation: "Rewording problems by making the semantic relations more explicit compensates for the less developed semantic schemata and facilitates appropriate bottom-up processing."

Staub and Reusser (1992)

In one of our own studies, 52 first graders (average age 7;11) and 37 third graders (10;0) were presented with a subset of specifically reworded change problems. Out of the six types of change problems the two easiest (Change 1 and Change 2) and the two most difficult (Change 5 and Change 6) problems were chosen. These four problem types correspond to logical combinations of the identity of the unknown quantity (initial state vs. resulting state) by direction of transfer (transfer-in vs. transfer-out). In the following, we list the Change 1 and the Change 6 problems used by Riley and Greeno (1988) followed by an example (in italics) of a reworded version used in our study:

Change 1. Joe had 3 marbles.
Then Tom gave him 5 more marbles.
How many marbles does Joe have now?
Today Dane got 11 marbles from Susan.
Yesterday Dane found 5 marbles.
How many marbles does Dane have now?

Change 6. Joe had some marbles.
Then he gave 5 marbles to Tom.
Now John has three marbles.
How many marbles did John have in the beginning?
Peter has 4 apples now.
Today Peter gave Mary 7 apples.
How many apples did Peter pick yesterday?

To control material factors such as names, type of objects, numbers, and kind of transfer verbs used, the change problems were each instantiated in different material versions.

TABLE 16.2
Comparison of Proportion of Correct Solutions of Data from Staub and Reusser (1992) with Data from Riley and Greeno (1988)

	Proportions of Correct Solutions (Without Blocks)			
	Grade 1		*Grade 3*	
Problem Type	*(1)*	*(2)*	*(1)*	*(2)*
Change 1	1.00	.63	1.00	.69
Change 2	1.00	.31	1.00	.47
Change 5	.33	.10	.95	.15
Change 6	.39	.32	.90	.30

Note. (1) Data from Riley & Greeno (1988); Grade 1: $n = 18$, Grade 3: $n = 20$. (2) Data from Staub and Reusser (1992); Grade 1: $n = 54$, Grade 3: $n = 37$.

For each of the four problem types the proportion of correct solutions was calculated and compared to the data of Riley and Greeno (1988). Table 16.2 shows that our problems are more difficult. But, regardless of the additional variations in our word problems, we also clearly replicated a strong tendency that has been shown before (e.g., Cummins, Kintsch, Reusser, & Weimer, 1988; Riley & Greeno, 1988; Riley et al., 1983): Problems with an unknown resulting state are easier than problems with an unknown initial state.

However, in our data there is also an interaction between identity of the unknown quantity and direction of transfer, which cannot be accounted for by the cognitive models based on a task analysis that concentrates on the logico-mathematical structure. Problems requiring students to determine a resulting state are easier if it is a transfer-in task (Change 1), as compared to transfer-out tasks (Change 2); when an initial state has to be determined it is the transfer-out tasks (Change 6) that are more likely to be solved.

Stern and Lehrndorfer (1992)

In a study with 45 first graders (average age: 6;10), Stern and Lehrndorfer used compare problems, such as "Peter has 6 crayons. Laura has 4 crayons. How many crayons less does Laura have than Peter?" These problems were better solved if the problem statements followed a description of a situational context referring to a competitive situation that is compatible with the problem question, such as the following: "Peter is Laura's older brother. Because he is older, his bedroom is larger and his toys are more expensive than Laura's. Peter also gets more pocket money than Laura and he has a new bike whereas Laura has Peter's old bike. When Peter does his homework, Laura doodles a little bit." Problem statements following such a situational context were better solved than problems with a preceding incompatible or neutral context description. Stern and Lehrndorfer interpreted these findings as indicating that the difficulties with compare problems are not solely caused by abstract language expressions, such as "How many more . . .", but may also be attributed to a lack of appropriate situational and mathematical knowledge.

The standard problems used by Riley et al. (1983) and others show very little verbal and situational variation. By furthermore acknowledging the empirical effects produced by minor specific changes in wording—without changing the logico-mathematical or basic semantic structure of the problems—it is evident that the theories accounting for the relative difficulty of mathematical word problems may only be valid within a rather narrow task space, whose dimensions have not yet been specified clearly. Children's difficulties in solving mathematical word problems cannot be accounted for solely by a lack of abstract logico-mathematical knowledge.[1] As the results referred to previously demonstrate, (minor) variations in the wording of standard problems may have a significant impact on problem difficulty that cannot be explained by the abstract logico-mathematical knowledge structures.

How does wording relate to abstract logico-mathematical structures? How are we to explain the relevant structural differences that have been captured by such colloquial notions as wording? In what follows, we present a theoretical framework that hopefully will be heuristically fruitful for looking at wording effects from a perspective that will allow the search for logico-mathematical schemata to be integrated with the analysis of situational structure and linguistic surface structure. We think that a theoretical analysis of the notion of wording is necessary not only for further research but also in order to form a coherent picture of what has been learned so far about what makes elementary arithmetic word problems difficult.

A THEORETICAL FRAMEWORK FOR ANALYZING THE UNDERSTANDING AND SOLVING OF ARITHMETIC WORD PROBLEMS

Our theorizing is guided by the epistemological intuition that early mathematical learning and thinking are embedded in the development of acting, thinking, language comprehension, and qualitative world knowledge. Children are able, through their experience with everyday acting and problem solving with real objects, to behave in proto-mathematical ways a long time before they enter schooling and master the fine-grained mathematical language.[2] Piaget (1950)

[1] Other intriguing findings have been collected by looking at "street math": It has been demonstrated that performance on mathematical problems embedded in real-life contexts was superior to that on school-type word problems and context-free computational problems involving the same numbers and operations (T. N. Carraher, D. W. Carraher, & Schliemann, 1985).

[2] Moreover, accumulating research shows that children have significant implicit understanding of counting, numbers, and sets before they enter formal schooling (Gelman & Greeno, 1989; Resnick, 1989). It has even been shown—by using habituation techniques—that 6- to 8-month-old infants have some sensitivity to numerosity (Starkey, Spelke, & Gelman, 1990). There is evidence that children as young as 3 or 4 years of age have what Resnick (1989) called "implicit protoquantitative reasoning schemas" for interpreting changes as increase or decrease as well as a protoquantitative "part-whole schema."

claimed that mathematical thinking emerges from acting. Mathematical operations are the developmental derivatives of sensorimotor actions, or, as Aebli (1980) put it from his broader cognitive instructional perspective, operations are "abstract actions." Concrete actions, referred to, for example, by the use of action verbs (giving, getting, selling, or losing a set of objects) are seen as bearing an abstract proto-mathematical meaning, a relational core that can be expressed and formalized by abstract mathematical operation schemata (such as adding, subtracting, multiplying, or dividing).

The Psychological Process Model

Extending the work of van Dijk and Kintsch (1983) and Kintsch and Greeno (1985), Reusser (1985, 1989b, 1990) developed a cognitive simulation model of understanding and solving elementary word arithmetic problems called the "Situation Problem Solver" (SPS).[3] The model is based on a decomposition analysis of the language and situation comprehension skills. Its purpose is to model the problem-solving process by emphasizing its language and situation comprehension components.

In SPS the cognitive process of solving mathematical word problems is considered to be a strategic process from text to situation to equation, an elaborative and incremental process of comprehension of the problem situation denoted by the problem text, which is modeled as a stepwise transformation of the initial textual representation of the problem into an equation.

SPS is a rule-based model implemented in LISP. It takes as input elementary addition and subtraction word problems, and "understands" and solves them by means of various types of lexical, syntactic, semantic, and pragmatic (macro)strategies. These strategies are related to four mutually constraining levels of comprehension or problem representation that are constructed (see Fig. 16.1):

1. Text comprehension in SPS refers to the construction of a *textbase* (Kintsch, 1974), a propositional representation of the textual input (see also Perfetti & Britt, chap. 2 in this vol.).

2. Situation comprehension involves the construction of an *episodic situation model*, or mental model, of the situation denoted by the text. This step is achieved through application of comprehension strategies to the textbase, which generate an analysis of the temporal and functional structure of the situations and actions depicted in the problem texts (see also Graesser & Zwaan, chap. 7 in this vol.).[4]

[3] So far its implementation is limited to a broad variety of change problems.

[4] With respect to our flexibly worded change problems (transactions of objects between a variable number of co-actors), this means that action-analytic strategies search the problem texts for the initial state, the resulting state, and the direction of transfer. To understand the structure of an action situation in SPS means to figure out the temporal order and the direction of the events: knowing what leads to what in an action sequence.

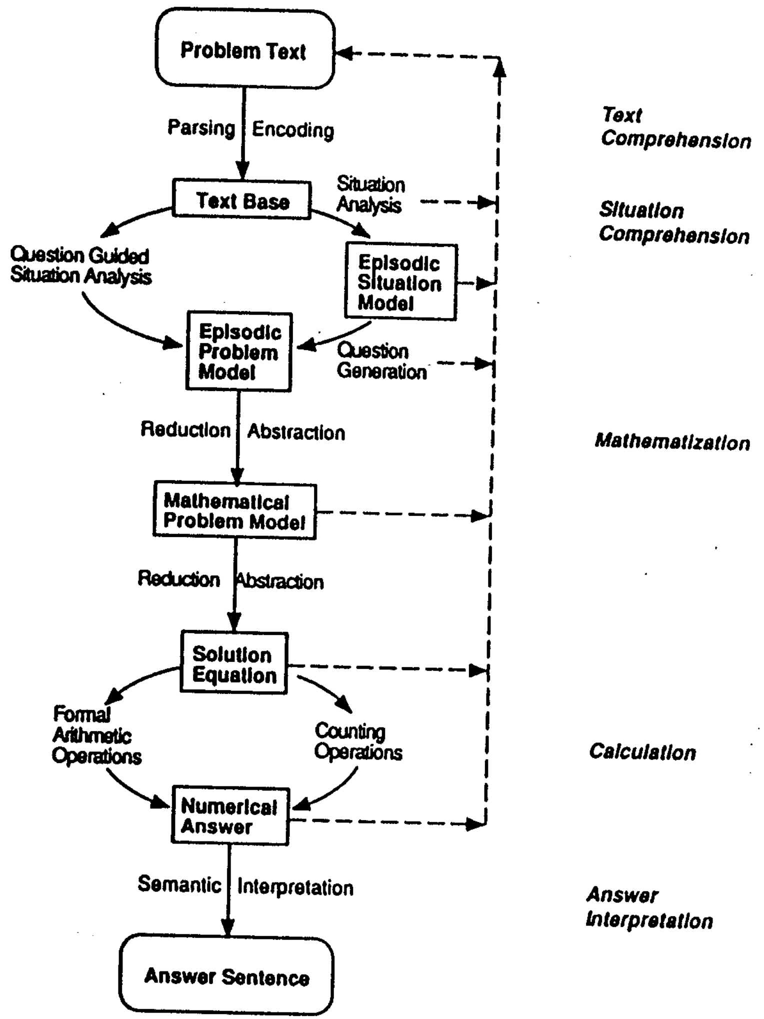

FIG. 16.1. Situation problem solver (SPS): Architecture and macrostrategies (Reusser, 1989b).

3. By constructing the *problem model*, mathematization in a broad sense sets in. A problem model includes all the structural elements and relations of the episodic situation model that are relevant from the point of view of the question to be answered. A meaningful question presupposes that by reflecting on the relevant conceptual and/or causal relations in the episodic situation model the answer can be determined. Such questions are mostly expressed by explicitly

asking for a quantitative answer.[5] By abstracting from situational specificity the problem model arrived at may already be a rather abstract semantic structure.

4. In the fourth phase, the problem model as a structure with a specified quantitative gap is further abstracted and reduced to its abstract mathematical gist, which leads to the *mathematical problem model*. The densest form of representing a structure of mathematical relations underlying a problem model is the numerical solution equation.

Finally, SPS generates an answer sentence referring back to the situation model and giving semantic meaning to the numerical answer.

In SPS the crucial step in the process of mathematization is the creation of a mental model and its reduction to the abstract mathematical gist. An (episodic) situation model, as a model of the real-world action or situation structure denoted by a problem text, can vary a great deal in elaboration and concreteness, and can be processed under very different operative perspectives.[6] In contrast, the problem model implies a quantitative operative perspective. To finally build a mathematical problem model means to see the situation exclusively in the light of a quantitative processing goal. The problem model is a special case of a situation model: It may still be concrete in many ways, but it contains—if it turns out to be suitable for successful mathematization—all the problem-relevant information needed for quantification.[7]

Based on this view of the levels of representation involved in understanding and solving a mathematical word problem for a given specific word problem, we may look for specific structural characteristics of the mental models to be constructed for a successful solution. For example, the following characteristics have to be distinguished:

[5]An explicit problem question asking for some numerical entity is only one way to communicate that quantification is to be the processing goal. A mathematical processing perspective can also be communicated by the semantic content of a story without an explicit question. For example, a protagonist expresses a goal that can only be successfully attained by mathematizing certain aspects of the story context (Aebli, Staub, & Ruthemann, 1991). In other words, the pursuit of many goals in everyday life "naturally" leads to mathematical acting, quantification, and calculation. In school settings, the operative perspective of quantification is conveyed to students also—if not mainly—by the situational context in which a task is presented—no matter how (un)meaningful the semantics. Rending a story problem in a mathematics class activates strong expectations about the kind of questions that might be asked by the teacher, or possibly should be asked by the student.

[6]One can think about a story's content in very different ways (aesthetical, motivational, causal, stylistic, etc.).

[7]For the parsimoniously worded "standard problems" it is in fact hardly possible to distinguish between situation and problem models because all the statements referring to the situation model are also part of the problem model. For most traditional math textbooks it is a characteristic feature of arithmetic word problems that situation and problem model overlap almost entirely (see also Aebli et al., 1991).

Situation model: static versus dynamic situation(s), number and type of co-actors involved, direction of transfer(s), and so on.

Problem model: identity of the unknown, "with" versus "without" an explicit action goal whose attainment requires mathematization of the situation.

Mathematical problem model: type of mathematical operation(s), given numerical quantities.

SPS is not a developmental model in the sense of Riley et al. (1983), or of Briars and Larkin (1984). The theoretical power of SPS rests in its implementation of analytic knowledge, allowing the simulation of language and situation comprehension processes while successfully solving a wide range of flexibly worded change problems. The basic psychological and instructional hypothesis associated with the idea of intervening episodic situation and problem models in SPS is that, for most problem solvers, situational understanding based on problem texts is not and, in an instructional context, should not be a superfluous but rather an obligatory outcome. The theoretical and empirical issue is that the logico-mathematical knowledge is merely one (though a very important) constraining factor in arriving at the right mathematical operation. Other important and probably underestimated factors that constrain the understanding and solving of problems are the underlying situations themselves and their linguistic expression or wording. That is, problems differ in the explicitness not only of the (mathematical) problem structure stated, but also in the quality of the problem question present (if at all); in the quality and degree of elaboration, coherence, and completeness of the situation description; in the sequential order of mention of situation elements; and in all kinds of presuppositions (drawing on a variety of world knowledge) implied by the verbal and situational setting.

The Presentational Structure of Arithmetic Word Problems

The facts and events that constitute the "world" referred to in a story text, such as the relations among the events of the story (temporal order, relations of causation, motivation), may be verbally described in many different ways. Following Morgan and Sellner (1980), we used the notion of *presentational structures* to refer to such variations as "the storyteller's choice concerning which points of content to present explicitly and which to leave to the hearer to infer, what order events should be presented in" (p. 185). To recover such structures entails inferring the writer's plan for presenting the story and determining how all the choices involved contribute to reaching the goal that motivates the plan.[8]

[8]Morgan and Sellner (1980) not only distinguished between a story's content and its presentational structure, they further spoke of a story's linguistic form, by which they referred to the linguistic elements and relations that make up the means used to express the story. A story's content, presentational structure, and linguistic form are not entirely independent. In discussing questions of linguistic form we refer to them as presentational variations.

The presentational structure of word problems thus refers to the manner in which a specific content is presented by use of specific linguistic means, following (at least implicitly) specific plans that ought to be tuned to the addressees and to the content to be conveyed. This conceptualization of presentational structure analytically presupposes an explication of a word problem's content. Once such a content structure is given, we can ask for possible presentational variations.[9]

By combining the notion of presentational structures with our process model, which distinguishes different levels of comprehension, we hope to gain a clearer view of the multitude of possible structural and presentational variations of mathematical word problems.

In arithmetic word problems, the abstract mathematical structure is of course not explicitly represented in the problem text. Except for the given quantities, the mathematical structure is intentionally hidden; its (re)construction is what constitutes an essential subgoal in order to produce answers to mathematical word problems as used in school settings. A given mathematical structure, posed as an applied mathematical problem, must be situationalized—that is, it must be presented with reference to a specific situational structure. Given that the instructional process aims at specific mathematical structures, then referring to a specific "reality" or situation to be mathematized constitutes the first basic presentational decision. Once a specific situational structure has been specified, one may further ask for possible variations of its presentational structure. For example, in the case of change problems, decisions concerning the presentation of situation and problem models would include text order (e.g., in relation to the "natural" sequence of action); narrative point of view in the episode(s); presence or absence of an explicit question; narrative point of view of the story question; explicitness of relevant relations (necessary inferences); specific lexical constructions referring to an abstract problem model; and manipulative material, pictures, diagrams, and so forth presented with the problem.

The studies using standard addition and subtraction word problems have almost exclusively varied features related to the structure of the problem model (that is, basic semantic structure and identity of the unknown) while limiting presentational variations.

THE EFFECTS OF WORDING REANALYZED

Based on the theoretical framework already presented, we will reanalyze some examples of the reworded standard problems that have been shown to produce effects of wording on solution rates.

We begin by looking more closely at the examples in the Hudson (1983) study. Hudson's more difficult question format, using "More," is indeed quite

[9]Such presentational variations in the recall of a given problem text can be used as valuable indicators for diagnosing comprehension (Staub, 1991).

abstract: "Here are some birds and here are some worms. How many more birds than worms are there?" Whereas the problem in the "Won't Get" format, which is also subsumed under the same logico-mathematical structure of the Compare 1 type, is richer in its wording: "Here are some birds and here are some worms. Suppose the birds all race over, and each one tries to get a worm. Will every bird get a worm? . . . How many birds won't get a worm?" Hudson explained the difficulties with the "More" question format to be a matter of linguistic form, in that the children are not able to correctly interpret the meaning of "more" in the question of the problem.

Gelman and Greeno (1989) considered Hudson's result as a demonstration of the effects of wording on children's performance: "It seems that children are able to compare the sets by forming sets with one-to-one correspondence and counting the remainder, when enough linguistic cues are provided" (p. 149). They explained the difficulties of the "More" questions as ignorance of the *principle of linguistic set difference* on the part of the children, that is, the missing linguistic knowledge that quantifiers (e.g., numerals or expressions, such as "some" or "how many") not only denote the cardinality of sets (linguistic cardinality) but may also denote the numerical difference between sets.

This may be a good characterization of the missing knowledge of children who do have difficulty solving the abstractly worded "More" problem. Yet, we think it is also important to ask why the easier "Won't Get" format is successfully solved. The explanation that "enough linguistic cues" are provided, in our view, does not state clearly enough the importance of the underlying situational structure and its presentational structure. The "Won't Get" task refers to a very different situation model. Whereas the question in the "More" task refers to an abstract, static situation, the question in the "Won't Get" format refers to a familiar situation or script that imbues the difference set with real-world meaning. In the "Won't Get" condition, the text is about some familiar action and motivational context (birds racing for worms, people eating cookies, kids picking up bikes). The text sequence follows the natural order of events and the question refers to an aspect of the outcome of the action being described: for example, the cardinality of a set referring to the birds left without any worms. Only the abstract problem model of the "Won't Get" task is the same as in the "More" task. That is, the two problems do not differ with respect to their abstract logico-mathematical structure: In both problem formats it is the difference set that is to be determined. In the "More" task condition the episodic situation model consists of two disjoint sets of entities without any further real-world semantics. The question solely introduces a very abstract relation between the two sets of entities by directly asking for the difference between the cardinalities of the two disjoint sets. Thus, the two task conditions in Hudson's study differ not only in the linguistic form of the question; the two conditions are moreover confounded with differences in the underlying episodic situation models. In terms of Piaget's and Aebli's cognitive developmental framework of mathematical concepts and operations, only the "Won't Get" task makes contact with the everyday proto-mathematical experiences, with familiar action scripts (with underlying motives of their actors and protagonists), or, with the concrete-operational roots in which the understanding of elementary mathematical operations is ultimately grounded—an understanding that gradually evolves long before children enter formal schooling.

As a second example, we again look at a reworded word problem as investigated by De Corte et al. (1985): "Peter and Mary have 8 nuts altogether. Five *of these* nuts belong to Mary. *The rest belong to Peter.* How many nuts does Peter have?"

Compared to its standard version ("Peter and Mary have 8 nuts altogether. Five nuts belong to Mary. How many nuts does Peter have?") the problem, as presented by De Corte et al., has been transformed by adding additional words (in italics). In the previous example, the rewording does not change the underlying situation model, the problem model, or the mathematical structure. It is exclusively the presentational structure of the situation and problem model that is altered by varying the amount of explicitly provided information about relations between elements of the problem model: In addition to the cardinality of two sets, specified by their owners, the relations between these two sets are further explicated. This additional wording is in fact not only helpful, but necessary in order to unambiguously generate the problem model. That is the case unless we assume the reader already has the presupposition that no object can be owned by more than one person at a time, except in an abstract kind of joint ownership referring to the superset of disjoint, privately owned sets (marked by phrases such as "have . . . altogether"). If the relations between the sets of objects are more explicitly and redundantly stated, adequate understanding does not depend as much on understanding single expressions, such as the two different meanings of "have" in the earlier problem text. The more explicit presentational version of the problem is thus less likely to lead to misunderstandings.

Variations in wording of standard problems analyzed by Staub and Reusser (1992) have shown that the following combined variations in presentational structure lead to a dramatic increase in difficulty:

1. The initial state has not been specified as a static state of possession (e.g., "Joe had 3 marbles"). Instead a minor variation in the situational structure was introduced by referring to an action by means of a transfer-in verb (e.g., "Joe collected 3 marbles"; "How many cookies did Tom bake yesterday?"). From this information it has to be inferred, by drawing on general world knowledge about transfer of objects and possession, that the outcome of this action leaves its protagonist as the possessor of the objects involved.[10]

[10]This inference corresponds to a special case of an addition problem. An initial state of zero objects of a certain kind is increased by some transfer-in. In this respect, our more complex situation corresponds to a two-step problem with the first operation always corresponding to the trivial operation: $0 + x = x$.

2. The text sequence of problem versions analyzed by Staub and Reusser does not match the *ordo naturalis* of the events as they would occur in the real world. Instead, the following sequences were used: resulting state/transfer/initial state for Change 5 and Change 6 problems, and transfer/initial state/resulting state for Change 1 and Change 2 problems. Our revision consisted of adding the time adverbs "yesterday," "today," and "now," together with appropriate tenses, in order to signal the temporal structure of the episodic elements.

3. In the standard problems the transfer of the objects between two persons is always denoted by the action verb "give" (e.g., "Tom had 8 marbles. Then he gave 5 marbles to Joe."). For problem models depicting a transfer-in, this requires the co-actor to take the position of the grammatical subject: "Tom had 3 marbles. Then Joe gave him 5 marbles." Contrary to this way of linguistically presenting the situation, in Staub and Reusser we used a presentational structure that keeps the protagonist (the subject, whose quantity of objects is of interest) in grammatical subject position. The narrative perspective thus remains constant, which should make it easier to comprehend (cf. Black, Turner, & Bower, 1979; Reusser, 1989a). In order to refer to the same problem model, this manipulation, on the other hand, requires the main transfer verb to be changed to "get"; for example, "Today Dane got 11 marbles from Susan. Yesterday Dane found 5 marbles."

4. In the standard problems, one of the persons involved in the transaction of the objects is pronominalized (for transfer-out problems it is the protagonist, for transfer-in it is the co-actor). This requires a bridging inference in order to connect the pronoun to its referent, a problem we eliminated by repeating the proper names throughout.

We speculate that of all these variations in the presentational structure, the dramatic increase in problem difficulty is mainly caused by the variation in text order (Variation 2), which makes it difficult for children to recover the intended situational structure (see also Ohtsuka & Brewer, 1992).

In Stern and Lehrndorfer (1992), a broad elaboration of the situation model that is compatible with the problem question and hence with the problem model—without any further changes in the presentational structure of the standard problem statements—proved to make the corresponding problem solution easier.

As this brief analysis of a few examples of wording effects demonstrates, the quality of the changes in wording, which have all been shown to affect problem difficulty, are quite diverse in character. The theoretical framework already presented, we think, will make it possible to specify much more clearly what kind of differences in wording are being compared in future studies.

CONCLUSIONS AND EDUCATIONAL SIGNIFICANCE

We first make some comments on theoretical questions and then argue, from an instructional point of view, why we think it is important to look at the presentational structure of mathematical word problems.

Theoretical Questions

The theoretical base of many studies using the standard Riley–Heller–Greeno problems has led to a focus mainly on differences in the underlying logico-mathematical problem structure. Problems with the same underlying logico-mathematical structure are seen as being isomorphic. However, as the empirical evidence shows, there are other factors that easily destroy this kind of problem isomorphism.

Our main focus has been on further characteristics of arithmetic word problems whose impact on problem difficulty has often been called "effects of wording." Within our conceptual framework of a cognitive task analysis, which focuses on the presentational structure of the situation and problem models of arithmetic word problems, we propose to theoretically disentangle the fuzzy colloquial notion of wording. Problem difficulty is analyzed as a complex interaction between semantic content structures (including textbase, situation model and problem model), the underlying mathematical structure, and the surface structures by which these are presented to a reader. There can be no simple theory of problem isomorphs: Problems can be similar (isomorphic) or different with regard to more than one level of structural descriptions. The solving of mathematical word problems is a language and knowledge intensive undertaking and should be seen as a skillful interaction of text comprehension (linguistic knowledge), situation comprehension (world-knowledge), and mathematical comprehension (mathematical knowledge).

Instructional Objectives

Examining the presentational structure of verbal problems is of considerable significance not only for the psychological explanation of the relative difficulty of mathematical word problems, but also for the design of such problems. What we refer to, from a psychological point of view, as presentational structures relates from an instructional point of view to a theory of pedagogical design (see Fig. 16.2). Textbook authors (e.g., in mathematics education) intuitively or consciously draw on knowledge about what makes problems (or texts) difficult and pedagogically valuable when constructing mathematical word problems. Writers are guided by certain intentions in selecting content, presentational structure, and linguistic devices (Clark, 1985). In looking at word problems we should ask the same questions: What do we know about the intentions, reasoning, and strategies of teachers and textbook authors that lead them to select a specific content, to give it a specific presentational structure? What do authors of mathematics textbooks have in mind when they choose or construct a mathematical word problem?

Teachers as well as textbook authors design word problems by drawing primarily on their intuitions about what kind of wording will be appropriate, for

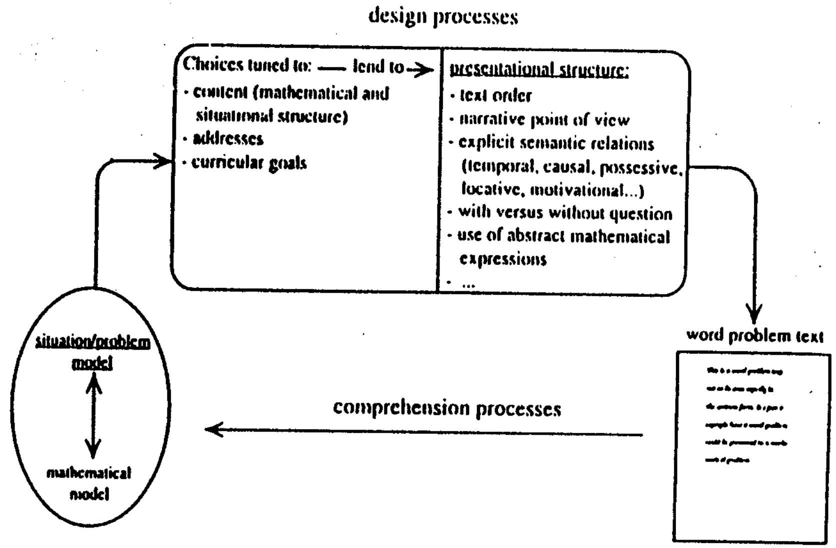

FIG. 16.2. Analyzing the design of mathematical word problems in terms of content (mathematical structure and situation model) and presentational structure, which both must be adjusted to addressees and curricular goals.

example, what expressions will signal as clearly as possible or with a presumed moderate difficulty the intended arithmetic operation structure.

From an instructional point of view, we need to inquire about the goals, purposes, and plans that are related to the presentational structure of specific word problems. We think that educational psychology can help improve instructional design by contributing to educator's knowledge about how the variation of presentational structures affects both problem difficulty and solution strategies. This instructional knowledge could provide criteria for selecting or generating problem texts whose instructional objectives are to foster students' specific knowledge and skills in comprehending textually presented situations that are to be mathematized. Thus, they contribute to students becoming flexible discourse and problem comprehenders. Because word problems have a clear final processing goal and because a successful solution largely depends on thoroughly understanding the situations denoted in a text, these problems provide excellent opportunities to explicitly apply world knowledge, discourse and language knowledge, as well as arithmetic knowledge.

Word problems should be analyzed with regard to all types of competencies that are required to solve them: language, situation, and mathematical comprehension. Thus, when using word problems in educational settings, the instructional goals should be directed at all these different factors. Word problems are a text type that allows and requires students to bridge the gap between the (textually represented) world of situations and the more abstract conceptual world of mathematics. If mathematics is not only to be taught as a kind of formal game for an elite population, learning to apply mathematical knowledge and principles to real situations ought to be a very important instructional goal.

The objectives being pursued with mathematical word problems should not lie exclusively in the field of mathematics education, but should also include objectives of language education and even content-related topics in science education and the humanities. Word problems should clearly be seen as exercises in applied mathematical thinking, as opportunities for exercising and reflecting on language comprehension skills, and even as opportunities to exercise some general problem-solving skills—such as planning, which is essential for solving complex mathematical story problems (Staub, 1988).

We postulate, from an educational point of view—in parallel with our psychological processing perspective—that the ultimate instructional goal associated with the use of "applied" mathematical problems is to foster and strengthen the relationship or connection between language education, subject matter education, and mathematics education.

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